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THE FORMATION OF A LIMIT CYCLE AT A "MERGED FOCUS"

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Formulas are presented which make it possible to trace the formation of a limit cycle at a merged focus using coefficients in the right-hand sides of dynamic systems.

In merged dynamic systems specified in the phase plane by different analytic expressions on opposite sides of some line in that plane [1], the "merging line" may contain points whose neighborhoods are similar to those of the equilibrium states of smooth systems. One of such points is a merged focus whose spirals consist of arcs lying on different sides of the merging line. The merged focus, like that of the smooth system, may change its stability when parameters of the system are changed. And, as in the case of the smooth system, the stability region boundary in the parameter space can be either "safe" if at transition through it from the stability to the instability region a stable limit cylce is originated at the focus, or "unsafe" when an unstable cycle contracts to it. If it is possible to find general integrals for merged systems, the form of the boundary is determined by the method presented in [2]. Below we describe a procedure which uses for this purpose only several first terms of expansions in series of the right-hand sides of merged systems.

Let us consider a system consisting of two analytic systems

$$x^{\cdot} = P_1(x, y), \quad y^{\cdot} = Q_1(x, y), \quad x > 0$$
$$x^{\cdot} = P_2(x, y), \quad y^{\cdot} = Q_2(x, y), \quad x < 0$$

merged on the y-axis. Conditions

$$P_1(0, 0) = P_2(0, 0) = 0, \quad Q_1(0, 0)\, Q_2(0, 0) < 0$$
$$\frac{\partial P_1(0,0)}{\partial y} Q_1(0,0) < 0, \quad \frac{\partial P_2(0,0)}{\partial y} Q_2(0,0) > 0$$

ensure that the structure in the neighborhood of the coordinate origin is of the merged focus type and, also, isolate the basic case in which the point (0, 0) does not represent the equilibrium state for any of the merged systems, as well as that the isoclines of the system vertical slopes reach the coordinate origin without being tangent to the y-axis.

These conditions make it possible to represent the system as

$$x^{\cdot} = a_1 x + a_2 y + P_1'(x, y), \quad y^{\cdot} = b + Q_1'(x, y), \quad x > 0$$
$$x^{\cdot} = c_1 x + c_2 y + P_2'(x, y), \quad y^{\cdot} = d + Q_2'(x, y), \quad x < 0$$
$$(a_2 b < 0, \; c_2 d > 0, \quad bd < 0)$$

where functions P_1', Q_1', P_2' and Q_2' can be expanded in series beginning with terms of an order higher than stated.

After the substitution

$$x^\circ = -(b/a_2)x, \quad y^\circ = (a_1/a_2)x + y, \quad x > 0 \tag{1}$$
$$x^\circ = (d/c_2)x, \quad y^\circ = (c_1/c_2)x + y, \quad x < 0$$

the system assumes the form

$$x^\cdot = -by + P_1''(x, y), \quad y^\cdot = b + Q_1''(x, y), \quad x > 0$$
$$x^\cdot = dy + P_2''(x, y), \quad y^\cdot = d + Q_2''(x, y), \quad x < 0$$

where functions P_1'', Q_1'', P_2'' and Q_2'' can be expanded in series beginning with terms of an order higher than stated, and indices at new variables have been omitted.

Reducing each system to a single equation and expanding the right-hand sides in series, we obtain

$$dx/dy = -y + g_{20}x^2 + g_{11}xy + g_{02}y^2 + \ldots + g_{12}xy^2 + g_{03}y^3 + \ldots + g_{04}y^4 + \ldots, \quad x > 0 \tag{2}$$

$$dx/dy = y + h_{20}x^2 + h_{11}xy + h_{02}y^2 + \ldots + h_{12}xy^2 + h_{03}y^3 + \ldots + h_{04}y^4 + \ldots, \quad x < 0$$

where only terms whose coefficients appear in final formulas are shown.

Let us consider the pointwise mapping $y_1 = y_1(y_0)$ (see Fig. 1) of the negative semiaxis y into the positive one using trajectories of the "right-hand" system and the mapping $y_2 = y_2(y_0)$ using trajectories of the "left-hand" system. To construct these images we solve Eqs. (2) which begin on the y-axis for small in absolute value $y = y_0$, representing these in the form of series in powers of y and y_0. In the obtained solutions we set $x = 0$, and obtain equations which we use for determining functions $y_1(y_0)$ and $y_2(y_0)$ in the form of series in powers of y_0

$$y_1 = -y_0 + {}^2/_3 g_{02}y_0^2 + \ldots + {}^2/_{135}(40\, g_{02}^3 + 45\, g_{02}g_{03} + 9g_{11}g_{02} + 18\, g_{20} + 9g_{12} + 27\, g_{04})\, y_0^4 + \ldots$$

$$y_2 = -y_0 - {}^2/_3 h_{02}y_0^2 + \ldots + {}^2/_{135}(-40\, h_{02}^3 + 45\, h_{02}h_{03} - 9h_{11}h_{02} - 18h_{20} + 9h_{12} - 27h_{04})\, y_0^4 + \ldots$$

We compose the remainder

$$f(y_0) \equiv y_1(y_0) - y_2(y_0) = {}^2/_3\, \alpha_2 y_0^2 + \ldots + {}^2/_{135}\, \alpha_4 y_0^4 + \ldots$$

where the coefficients at $y_0^2, \ldots, y_0^4, \ldots$ are the remainders of related coefficients of series y_1 and y_2.

Simple geometric considerations show that function $f(y_0)$ is of the same sign for $y_0 > 0$ and $y_0 < 0$. Hence the ordinal number of the first nonzero coefficient is even. If it is positive, then for small $|y_0|$ we have $f(y_0) > 0$ and the focus is stable, while when it is negative, the focus is unstable. This holds for clockwise motion $(d > 0)$ around the focus; when the motion is in the opposite direction $(d < 0)$ the reverse is true. When all coefficients of the series $f(y_0)$ are zero, $f(y_0) \equiv 0$ and the equilibrium state is a "merged center".

If we now assume that the system depends on parameters (the right-hand sides of both merged systems are analytic functions of x, y, and of parameters), then the coefficients of series $f(y_0)$ are functions of these parameters. Since a change of the sign of α_2 results in the change of stability, the condition $\alpha_2 = 0$ determines in the parameter space a surface which is the stability region boundary (at the boundary both

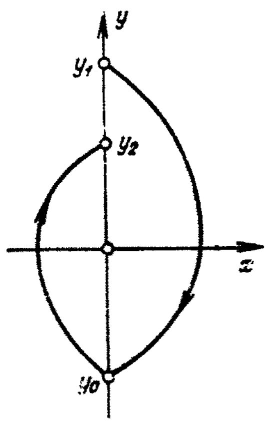

Fig. 1

α_2 and α_3 vanish).

Let $\alpha_4 \neq 0$ correspond to parameters at the boundary itself. The equilibrium state of such parameters is determined by a focus whose stability depends on the sign of α_4 and the direction of motion along trajectories. If the focus is stable, then the boundary is safe, as in the case of a smooth system, and when it is unstable, the boundary is unsafe. The proof of these statements is similar to that appearing in [3] in the case of an analytic system.

Thus the determination of the stability region boundary and of its form involves the following: 1) carrying out transformation (1); 2) substituting a single equation for each merged system and expand their right-hand sides in series, retaining only the terms that appear in (2), and 3) determining $\alpha_2 = g_{02} + h_{02}$

$$\alpha_4 = (5g_{02}g_{03} + g_{11}g_{02} + 2g_{20} + g_{12} + 3g_{04}) - (5h_{02}h_{03} - h_{11}h_{02} - 2h_{20} + h_{12} - 3h_{04})$$

where the evident simplifications are justified by that α_4 is only required when $\alpha_2 = 0$,

The stability region boundary is determined by the equality $\alpha_2 = 0$ and the region of stability (instability) is determined by the inequality $d\alpha_2 > 0$ (< 0); when at the boundary $d\alpha_4 > 0$ (< 0), it is safe (unsafe).

E x a m p l e. Let us consider an on-off servomechanism working under conditions of constant speed of the input shaft [4].

The equations of motion are

$$x^{\cdot\cdot} = -M(\omega - x^{\cdot}),\ x > 0; \quad x^{\cdot\cdot} = M(x^{\cdot} - \omega),\quad x < 0$$

where $M(v)$ is the mechanical characteristic of the motor (dependence of torque on velocity v), $\omega > 0$ is the input shaft speed, and x is the mismatch.

We approximate the characteristic of the motor by the cubic parabola

$$M(v) = mv^3 + nv^2 + pv + r,\quad m < 0,\quad n < 0,\quad p > 0,\quad r > 0$$

The servomechanism that corresponds to this equation is defined by

$$x^{\cdot} = y,\quad y^{\cdot} = -m(\omega - y)^3 - n(\omega - y)^2 - p(\omega - y) - r,\quad x > 0$$

$$x^{\cdot} = y,\quad y^{\cdot} = m(y - \omega)^3 + n(y - \omega)^2 + p(y - \omega) + r,\quad x < 0$$

For not very high ω its equilibrium position $(0, 0)$ is at the merging line.

The stability region boundary in the parameter plane ω^2, p is defined by the curve

$$p = (mn\omega^4 + 3mr^2\omega^2) / (n\omega^2 - r)$$

Above the boundary line $d\alpha_2 < 0$ and the equilibrium state is represented by a merged unstable focus, below it $d\alpha_2 > 0$ and the focus is stable. At the stability region boundary

$$d\alpha_4 = -2(np + 3mr) / M(\omega) > 0$$

The boundary is safe.

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